

On the index of a free abelian subgroup in the group of central units of an integral group ring

Gurmeet K. Bakshi and Sugandha Maheshwary^{*†}

Centre for Advanced Study in Mathematics,

Panjab University, Chandigarh 160014, India

email: gkbakshi@pu.ac.in and msugandha.87@gmail.com

Abstract

Let $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ denote the group of central units in the integral group ring $\mathbb{Z}[G]$ of a finite group G . A bound on the index of the subgroup generated by a virtual basis in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ is computed for a class of strongly monomial groups. The result is illustrated with application to the groups of order p^n , p prime, $n \leq 4$. The rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ and the Wedderburn decomposition of the rational group algebra of these p -groups have also been obtained.

Keywords : integral group rings, unit group, central units, generalized Bass units, Wedderburn decomposition, strong Shoda pairs, strongly monomial groups, normally monomial groups.

MSC2000 : 16U60; 16K20; 16S34; 20C05

1 Introduction

Let $\mathcal{U}(\mathbb{Z}[G])$ denote the unit group of the integral group ring $\mathbb{Z}[G]$ of a finite group G . The centre of $\mathcal{U}(\mathbb{Z}[G])$ is denoted by $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$. It is well known that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm \mathcal{Z}(G) \times A$, where A is a free abelian subgroup of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ of finite rank. In order to study $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$, a multiplicatively independent subset of such a subgroup A , i.e., a \mathbb{Z} -basis for such a free \mathbb{Z} -module A , is of importance, and is known only for a few groups ([1, 2, 7, 18], see also [20], Examples 8.3.11 and 8.3.12). However, other papers deal with determining a virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$,

^{*}Research supported by CSIR, India, is gratefully acknowledged

[†]Corresponding author

i.e., a multiplicatively independent subset of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ which generates a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ (see e.g. [6, 9, 10, 11, 12, 13, 14, 15, 16, 17]).

Analogous to well known cyclotomic units in cyclotomic fields, Bass [4] constructed units, so called *Bass cyclic units*, which generate a subgroup of finite index in $\mathcal{U}(\mathbb{Z}[G])$, when G is cyclic. A virtual basis consisting of certain Bass cyclic units was also given by Bass. Generalizing the notion of Bass cyclic units, Jespers et al [13] defined *generalized Bass units* and have shown that the group generated by these units contains a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ for an arbitrary strongly monomial group G . Recently, for a class of groups properly contained in finite strongly monomial groups, Jespers et al [15] provided a subset, denoted by $\mathcal{B}(G)$ (say), of the group generated by generalized Bass units, which forms a virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$.

In this paper, we determine a bound on the index of the subgroup generated by $\mathcal{B}(G)$ in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ for the same class of groups as considered in [15] (Theorem 2). Our result is based on the ideas contained in [15] and Kummer's work (see [23], Theorem 8.2) on the index of cyclotomic units. Further in [15], Jespers et al have provided the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ in terms of strong Shoda pairs of G , when G is a strongly monomial group. In section 4, we compute a complete and irredundant set of strong Shoda pairs of the non abelian groups of order p^n , p prime, $n \leq 4$, and provide, in terms of p , the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ of these p -groups along with the Wedderburn decomposition of their rational group algebras. We also illustrate Theorem 2 for some of these groups.

2 Notation and Preliminaries

We begin by fixing some notation.

G	a finite group
$ g $	the order of the element g in G
g^t	$t^{-1}gt$, $g, t \in G$
$\langle X \rangle$	the subgroup generated by the subset X of G
$ X $	the cardinality of the set X
$K \leq G$	K is a subgroup of G
$K \trianglelefteq G$	K is a normal subgroup of G
$[G : K]$	the index of the subgroup K in G
$N_G(K)$	the normalizer of K in G
$\text{core}(K)$	$\bigcap_{x \in G} xKx^{-1}$, the largest normal subgroup of G contained in K

\hat{K}	$\frac{1}{ K } \sum_{k \in K} k$
$\mathcal{M}(G/K)$	the set of minimal normal subgroups of G containing K properly
$\varepsilon(H, K)$	$\begin{cases} \hat{H}, & \text{if } H = K; \\ \prod_{M/K \in \mathcal{M}(H/K)} (\hat{K} - \hat{M}), & \text{otherwise, where } K \trianglelefteq H \leq G \end{cases}$
$e(G, H, K)$	the sum of all the distinct G -conjugates of $\varepsilon(H, K)$
φ	Euler's phi function
$\mathbb{Z}/n\mathbb{Z}$	the ring of integers modulo n , $n \geq 1$
ζ_n	a primitive n^{th} root of unity in the field of complex numbers
$\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$	the Galois group of the cyclotomic field $\mathbb{Q}(\zeta_n)$ over \mathbb{Q}
h_n^+	the class number of the maximal real subfield of $\mathbb{Q}(\zeta_n)$
$\text{l.c.m.}(k, n)$	the least common multiple of the integers k and n
(k, n)	the greatest common divisor of the integers k and n
$o_n(k)$	the multiplicative order of k modulo n , where $(k, n) = 1$
$\eta_k(\zeta_n)$	$\begin{cases} 1, & \text{if } n = 1; \\ 1 + \zeta_n + \zeta_n^2 + \dots + \zeta_n^{k-1}, & \text{if } n > 1, \text{ where } k \geq 1 \end{cases}$
$\mathcal{U}(R)$	the unit group of the ring R
$M_n(R)$	the ring of $n \times n$ matrices over the ring R , $n \geq 1$
$M_n(R)^{(s)}$	$M_n(R) \oplus M_n(R) \oplus \dots \oplus M_n(R)$, direct sum of s copies, $s \geq 1$
I_n	the $n \times n$ identity matrix

A *strong Shoda pair* ([19], Definition 3.1) of G is a pair (H, K) of subgroups of G with the properties that

- (i) $K \trianglelefteq H \trianglelefteq N_G(K)$;
- (ii) H/K is cyclic and a maximal abelian subgroup of $N_G(K)/K$;
- (iii) the distinct G -conjugates of $\varepsilon(H, K)$ are mutually orthogonal.

Note that (G, G) is always a strong Shoda pair of G .

If (H, K) is a strong Shoda pair of G , then $e(G, H, K)$ is a primitive central idempotent of the rational group algebra $\mathbb{Q}[G]$ ([19], Proposition 3.3). A group G is called *strongly monomial* if every primitive central idempotent of $\mathbb{Q}[G]$ is of the form $e(G, H, K)$ for some strong Shoda pair (H, K) of G .

Two strong Shoda pairs (H_1, K_1) and (H_2, K_2) of G are said to be *equivalent* if $e(G, H_1, K_1) = e(G, H_2, K_2)$. A complete set of representatives of distinct equivalence classes of strong Shoda pairs of G is called a *complete irredundant set of strong Shoda pairs* of G . In case G is strongly monomial, one can calculate the

primitive central idempotents of $\mathbb{Q}[G]$ from a complete irredundant set of strong Shoda pairs of G .

Recall that a group G is called *normally monomial* if every complex irreducible character of G is induced from a linear character of a normal subgroup of G . Theorem 1, as stated below, provides an algorithm to determine a complete irredundant set of strong Shoda pairs of a normally monomial group G and also, in particular, yields that a normally monomial group is strongly monomial.

Let \mathcal{N} be the set of all the distinct normal subgroups of a finite group G . For $N \in \mathcal{N}$, set

- A_N : a normal subgroup of G containing N such that A_N/N is an abelian normal subgroup of maximal order in G/N .
- \mathcal{D}_N : the set of all subgroups D of A_N containing N such that $\text{core}(D) = N$, A_N/D is cyclic and is a maximal abelian subgroup of $N_G(D)/D$.
- \mathcal{T}_N : a set of representatives of \mathcal{D}_N under the equivalence relation defined by conjugacy of subgroups in G .
- \mathcal{S}_N : $\{(A_N, D) \mid D \in \mathcal{T}_N\}$.

Note that if $N \in \mathcal{N}$ is such that G/N is abelian, then, by ([3], Eq.(1)),

$$\mathcal{S}_N = \begin{cases} \{(G, N)\}, & \text{if } G/N \text{ cyclic;} \\ \emptyset, & \text{otherwise.} \end{cases} \quad (1)$$

Theorem 1 ([3], Theorem 1, Corollaries 1 and 2) *The following statements are equivalent:*

- (i) G is normally monomial;
- (ii) $\mathcal{S}(G) := \bigcup_{N \in \mathcal{N}} \mathcal{S}_N$ is a complete irredundant set of strong Shoda pairs of G ;
- (iii) $\{e(G, A_N, D) \mid (A_N, D) \in \mathcal{S}_N, N \in \mathcal{N}\}$ is a complete set of primitive central idempotents of $\mathbb{Q}[G]$;
- (iv) $|G| = \sum_{N \in \mathcal{N}} \sum_{D \in \mathcal{D}_N} [G : A_N] \varphi([A_N : D])$.

Let $n \geq 1$ and let k be an integer coprime to n . Then, $\eta_k(\zeta_n)$ is a unit of $\mathbb{Z}[\zeta_n]$. The units of the form $\eta_k(\zeta_n^j)$ with integers j, k and n such that $(k, n) = 1$ are called *cyclotomic units* of $\mathbb{Q}(\zeta_n)$.

Let $g \in G$ and let k, m be positive integers such that $k^m \equiv 1 \pmod{n}$, where $n = |g|$. Then,

$$u_{k,m}(g) = (1 + g + \dots + g^{k-1})^m + \frac{1 - k^m}{n}(1 + g + \dots + g^{n-1})$$

is a unit in the integral group ring $\mathbb{Z}[G]$. The units in $\mathbb{Z}[G]$ of this form are called *Bass cyclic units* (see [21], (10.3)).

Next, we recall the definition of *generalized Bass units* of $\mathbb{Z}[G]$ introduced by Jespers et al [13]. For $M \trianglelefteq G$, $g \in G$, and positive integers k, m such that $k^m \equiv 1 \pmod{|g|}$, let

$$u_{k,m}(1 - \hat{M} + g\hat{M}) = 1 - \hat{M} + u_{k,m}(g)\hat{M}.$$

This element is a unit in $\mathbb{Z}[G](1 - \hat{M}) + \mathbb{Z}[G]\hat{M}$. As both $\mathbb{Z}[G](1 - \hat{M}) + \mathbb{Z}[G]\hat{M}$ and $\mathbb{Z}[G]$ are orders in $\mathbb{Q}[G]$, there is a positive integer $n_{g,M}$ such that

$$(u_{k,m}(1 - \hat{M} + g\hat{M}))^{n_{g,M}} \in \mathcal{U}(\mathbb{Z}[G]). \quad (2)$$

Suppose $n_{G,M}$ is the minimal positive integer satisfying Eq.(2) for all $g \in G$. Then, the element

$$(u_{k,m}(1 - \hat{M} + g\hat{M}))^{n_{G,M}} = 1 - \hat{M} + u_{k,mn_{G,M}}(g)\hat{M}$$

is called the *generalized Bass unit* of $\mathbb{Z}[G]$ based on g and M with parameters k and m . Observe that $n_{G,M} = 1$, if M is trivial i.e., $M = \langle 1 \rangle$ or G .

Remark 1 For a non trivial normal subgroup M of G , using Lemma 3.1 of [8], it may be noted that $n_{G,M} = 1$, if every $g \in G \setminus M$ is of order 2; otherwise, $n_{G,M}$ is the minimal positive integer satisfying Eq.(2) for all $g \in G \setminus M$ with $|g| > 2$ and integers k, m such that $1 < k < |g|$, $(k, |g|) = 1$ and $m = o_{|g|}(k)$.

Let G be a strongly monomial group such that there is a complete irredundant set $\{(H_i, K_i) \mid 1 \leq i \leq m\}$ of strong Shoda pairs of G with the property that each $[H_i : K_i]$ is a prime power, say $p_i^{n_i}$. Assume that $(H_1, K_1) = (G, G)$. For such a group G , we recall the virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ provided by Jespers et al [15]. For $1 \leq i \leq m$, we adopt the following notation:

$$\begin{aligned} \varepsilon_i &:= \varepsilon(H_i, K_i) \\ e_i &:= e(G, H_i, K_i) \\ [H_i : K_i] &:= p_i^{n_i}, p_i \text{ prime, } n_i \geq 0 \text{ (} n_i = 0 \text{ only if } i = 1) \\ g_i K_i &:= \text{a generator of the cyclic group } H_i/K_i \\ L_j^{(i)} &:= \langle g_i^{p_i^{n_i-j}}, K_i \rangle, \ 0 \leq j \leq n_i \\ N_i &:= N_G(K_i) \\ m_i &:= [G : N_i] \\ T_i &:= \text{a right transversal of } N_i \text{ in } G \end{aligned}$$

For $2 \leq i \leq m$, consider the action of N_i/H_i on $\mathbb{Q}(\zeta_{p_i^{n_i}})$ given by the map

$$\begin{aligned} N_i/H_i &\longrightarrow \text{Gal}(\mathbb{Q}(\zeta_{p_i^{n_i}})/\mathbb{Q}) \\ n_i H_i &\longmapsto \alpha_{n_i H_i}, \end{aligned} \quad (3)$$

where $\alpha_{n_i H_i}(\zeta_{p_i^{n_i}}) = \zeta_{p_i^{n_i}}^j$, if $n_i^{-1} g_i n_i K_i = g_i^j K_i$. As H_i/K_i is a maximal abelian subgroup of N_i/K_i , it turns out that the above action is faithful. Hence, N_i/H_i is isomorphic to a subgroup of $\text{Gal}(\mathbb{Q}(\zeta_{p_i^{n_i}})/\mathbb{Q}) \cong \mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$. For the convenience of notation, we regard N_i/H_i as a subgroup of $\text{Gal}(\mathbb{Q}(\zeta_{p_i^{n_i}})/\mathbb{Q})$ and that of $\mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$. (Notice that N_i/H_i can be regarded as a subgroup of $\mathcal{U}(\mathbb{Z}/[H_i : K_i]\mathbb{Z})$, even if $[H_i : K_i]$ is not a prime power.) With this identification, N_i/H_i is equal to either $\langle \phi_{r_i} \rangle$ or $\langle \phi_{r_i} \rangle \times \langle \phi_{-1} \rangle$ (resp. $\langle r_i \rangle$ or $\langle r_i \rangle \times \langle -1 \rangle$) for some $r_i \in \mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$, where ϕ_{r_i} denotes the automorphism of $\mathbb{Q}(\zeta_{p_i^{n_i}})$ which sends $\zeta_{p_i^{n_i}}$ to $\zeta_{p_i^{n_i}}^{r_i}$. The later case arises only if $p_i = 2$ and $n_i \geq 3$. Set

$$d_i := \begin{cases} 1, & \text{if } -1 \in \langle r_i \rangle; \\ 2, & \text{otherwise.} \end{cases} \quad (4)$$

and

$$o_i := \begin{cases} 4, & \text{if } p_i = 2, N_i/H_i = \langle r_i \rangle, r_i \equiv 1 \pmod{4}, n_i \geq 2; \\ 6, & \text{if } p_i = 3, N_i/H_i = \langle r_i \rangle, r_i \equiv 1 \pmod{3}; \\ 2, & \text{otherwise.} \end{cases} \quad (5)$$

Further, choose a subset I_i of $\{k \mid 1 \leq k \leq \frac{p_i^{n_i}}{2}, (k, p_i) = 1\}$ containing 1, which forms a set of representatives of $\mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$ modulo $\langle N_i/H_i, -1 \rangle$. We extend the notation by setting $I_1 = \{1\}$, in view of the trivial action of the identity group N_1/H_1 on $\mathbb{Q}(\zeta_1) = \mathbb{Q}$.

Let k be a positive integer coprime to p_i and let r be an arbitrary integer. For $0 \leq j \leq s \leq n_i$, consider the following products of generalized Bass units of $\mathbb{Z}[H_i]$, defined recursively:

$$c_s^s(H_i, K_i, k, r) = 1$$

and, for $0 \leq j \leq s-1$,

$$\begin{aligned} c_j^s(H_i, K_i, k, r) &= \left(\prod_{h \in L_j^{(i)}} u_{k, o_{p_i^{n_i}}(k) n_{H_i, K_i}}(g_i^{r p_i^{n_i-s}} h \hat{K}_i + 1 - \hat{K}_i) \right)^{p_i^{s-j-1}} \times \\ &\quad \left(\prod_{l=j+1}^{s-1} c_l^s(H_i, K_i, k, r)^{-1} \right) \left(\prod_{l=0}^{j-1} c_l^{s+l-j}(H_i, K_i, k, r)^{-1} \right), \end{aligned}$$

where the empty products equal 1.

Define

$$\begin{aligned} B(H_i, K_i) &:= \{\prod_{x \in N_i/H_i} c_0^{n_i}(H_i, K_i, k, x) \mid k \in I_i \setminus \{1\}\}, \\ \mathcal{B}(H_i, K_i) &:= \{\prod_{t \in T_i} u^t \mid u \in B(H_i, K_i)\}, \end{aligned}$$

and

$$\mathcal{B}(G) := \bigcup_{i=1}^m B(H_i, K_i). \quad (6)$$

Jespers et al ([15], Theorem 3.5) proved that $\mathcal{B}(G)$ is a *virtual basis* of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$.

3 A bound on the index of $\langle \mathcal{B}(G) \rangle$ in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$

In this section, we continue with the notation developed in section 2.

Theorem 2 *Let G be a strongly monomial group and let $\{(H_i, K_i) \mid 1 \leq i \leq m\}$ be a complete irredundant set of strong Shoda pairs of G with $(H_1, K_1) = (G, G)$. For $2 \leq i \leq m$, let I_i be a subset of $\{k \mid 1 \leq k \leq \frac{[H_i:K_i]}{2}, (k, [H_i:K_i]) = 1\}$ containing 1, which forms a set of representatives of $\mathcal{U}(\mathbb{Z}/[H_i:K_i]\mathbb{Z})$ modulo $\langle N_i/H_i, -1 \rangle$, where $N_i = N_G(K_i)$. Set $I_1 = \{1\}$.*

- (i) *The rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = 0$ if and only if $|I_i| = 1$ for all i , $1 \leq i \leq m$, and in this case, $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm \mathcal{Z}(G)$.*
- (ii) *In addition, if $[H_i:K_i]$ is a prime power, say $p_i^{n_i}$, for all i , $1 \leq i \leq m$, and $\mathcal{B}(G)$ is the virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ as defined in Eq.(6), then,*

$$[\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) : \langle \mathcal{B}(G) \rangle] \leq 2 \prod_{\substack{i=2 \\ |I_i|=1}}^m o_i \prod_{\substack{i=2 \\ |I_i| \neq 1}}^m h_{p_i}^{+n_i} l_i p_i^{n_i-1} \mathfrak{o}_i(l_i^{d_i-1} [N_i: H_i])^{|I_i|-1},$$

where $l_i = l.c.m.(2, p_i)$; $\mathfrak{o}_i = \prod_{\substack{1 < k < \frac{p_i^{n_i}}{2} \\ (k, p_i)=1}} o_{p_i^{n_i}}(k) p_i^{n_i-1} n_{H_i, K_i}$; d_i and o_i are as defined in Eq.(4) and Eq.(5) respectively.

We first prove the following:

Lemma 1 *Let G be as in Theorem 2. Let $\mathcal{A}(H_i, K_i) = \mathcal{Z}(1 - e_i + \mathcal{U}(\mathbb{Z}[G]e_i))$ and $A(H_i, K_i) = \mathcal{Z}(1 - \varepsilon_i + \mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$, where $1 \leq i \leq m$. Then,*

$$[\mathcal{A}(H_i, K_i) : \langle \mathcal{B}(H_i, K_i) \rangle] = [A(H_i, K_i) : \langle B(H_i, K_i) \rangle].$$

Proof. Let $\{t_j \mid 1 \leq j \leq m_i\}$ be a right transversal of N_i in G with $t_1 = 1$. For $\alpha \in \mathbb{Q}[G]e_i$ and integers r and s such that $1 \leq r, s \leq m_i$, let $\alpha_{rs} = \varepsilon_i t_r \alpha t_s^{-1} \varepsilon_i$. We notice that $\alpha_{rs} \in \mathbb{Q}[N_i]\varepsilon_i$. To see this, write $\alpha = (\sum_{g \in G} \alpha_g g)e_i$ with $\alpha_g \in \mathbb{Q}$. Then $\alpha_{rs} = \sum_{g \in G} \alpha_g \varepsilon_i t_r g t_s^{-1} \varepsilon_i$. By ([19], Proposition 3.3), the centralizer of ε_i in G equals N_i . Therefore, if $t_r g t_s^{-1} \notin N_i$, then $\varepsilon_i t_r g t_s^{-1} \varepsilon_i = t_r g t_s^{-1} \varepsilon_i^{t_r g t_s^{-1}} \varepsilon_i = 0$. Also, if $t_r g t_s^{-1} \in N_i$, then $\varepsilon_i t_r g t_s^{-1} \varepsilon_i = t_r g t_s^{-1} \varepsilon_i$. Consequently, $\alpha_{rs} = \sum_{t_r g t_s^{-1} \in N_i} \alpha_g t_r g t_s^{-1} \varepsilon_i \in \mathbb{Q}[N_i]\varepsilon_i$.

Now consider the map

$$\theta_i : \mathbb{Q}[G]e_i \longrightarrow M_{m_i}(\mathbb{Q}[N_i]\varepsilon_i)$$

given by

$$\alpha \mapsto^{\theta_i} (\alpha_{rs})_{m_i \times m_i}.$$

As ε_i^t , $t \in T_i$, are mutually orthogonal idempotents and $\sum_{t \in T_i} \varepsilon_i^t = e_i$, it can be checked that θ_i is an isomorphism of \mathbb{Q} -algebras. This isomorphism in turn yields the group isomorphism

$$\mathcal{Z}(\mathcal{U}(\mathbb{Q}[G]e_i)) \stackrel{\theta_i}{\cong} \mathcal{Z}(\mathcal{U}(M_{m_i}(\mathbb{Q}[N_i]\varepsilon_i)))$$

given by

$$\alpha \mapsto^{\theta_i} \varepsilon_i \alpha \varepsilon_i I_{m_i}.$$

Let ψ_i denote the canonical isomorphism from $\mathcal{Z}(\mathcal{U}(\mathbb{Q}[N_i]\varepsilon_i))$ to $\mathcal{Z}(\mathcal{U}(M_{m_i}(\mathbb{Q}[N_i]\varepsilon_i)))$ given by $u \mapsto^{\psi_i} u I_{m_i}$. Denote by ϕ_i , the restriction of $\psi_i^{-1} \circ \theta_i$ to $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))$. We assert that ϕ_i is an isomorphism from $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))$ to $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$. For this, we need to show that

$$\phi_i(\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))) = \mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i)). \quad (7)$$

Consider $\alpha = (\sum_{g \in G} \alpha_g g)e_i \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))$ with $\alpha_g \in \mathbb{Z}$. We have $\varepsilon_i \alpha \varepsilon_i = \sum_{g \in N_i} \alpha_g g \varepsilon_i \in \mathbb{Z}[N_i]\varepsilon_i$. Consequently, $\phi_i(\alpha) = \varepsilon_i \alpha \varepsilon_i \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$, as we already have $\phi_i(\alpha) \in \mathcal{Z}(\mathcal{U}(\mathbb{Q}[N_i]\varepsilon_i))$. On the other hand, to see that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$ is contained in $\phi_i(\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i)))$, let $u \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$. Following the argument as in the proof of Theorem 3.5 of [15], it can be seen that $\sum_{t \in T_i} u^t$ belongs to $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))$, as ε^t , $t \in T_i$, are mutually orthogonal idempotents. One checks that $\sum_{t \in T_i} u^t$ maps to u under ϕ_i and hence Eq.(7) follows. The isomorphism ϕ_i now provides the group isomorphism

$$\Theta_i : \mathcal{A}(H_i, K_i) \longrightarrow A(H_i, K_i)$$

by setting

$$1 - e_i + \alpha \mapsto^{\Theta_i} 1 - \varepsilon_i + \varepsilon_i \alpha \varepsilon_i,$$

where $\alpha \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]e_i))$. We further see that if $u = 1 - \varepsilon_i + \gamma \in B(H_i, K_i)$, with $\gamma \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}[N_i]\varepsilon_i))$, then $\Theta_i(\prod_{t \in T_i} u^t) = \Theta_i(1 - e_i + \sum_{t \in T_i} \gamma^t) = 1 - \varepsilon_i + \varepsilon_i(\sum_{t \in T_i} \gamma^t)\varepsilon_i = 1 - \varepsilon_i + \gamma = u$. This yields $\Theta_i(\mathcal{B}(H_i, K_i)) = B(H_i, K_i)$ and consequently, Lemma 1 follows. \square

Lemma 2 *Let p be a prime and let $n \geq 1$ be an integer. For a subgroup A of $\mathcal{U}(\mathbb{Z}/p^n\mathbb{Z}) (\cong \text{Aut}(\langle \zeta_{p^n} \rangle))$, let $\mathcal{U}(\mathbb{Z}[\zeta_{p^n}]^A)$ denote the unit group of the fixed ring $\mathbb{Z}[\zeta_{p^n}]^A$. If $\langle A, -1 \rangle = \mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$, then*

$$\mathcal{U}(\mathbb{Z}[\zeta_{p^n}]^A) = \begin{cases} \langle \zeta_4 \rangle, & \text{if } p = 2, A = \langle r \rangle, r \equiv 1 \pmod{4}, n \geq 2; \\ \langle \zeta_6 \rangle, & \text{if } p = 3, A = \langle r \rangle, r \equiv 1 \pmod{3}; \\ \langle \zeta_2 \rangle, & \text{otherwise.} \end{cases}$$

Proof. If $\langle A, -1 \rangle = \mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$, then it follows from ([15], Lemma 3.2) that the rank of $\mathcal{U}(\mathbb{Z}[\zeta_{p^n}]^A) = 0$ and hence it consists only of the roots of unity. Recall that the group of roots of unity in $\mathbb{Q}(\zeta_{p^n})$ is $\langle \zeta_{p^n}, -1 \rangle$. Thus, in case $\langle A, -1 \rangle = \mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$, any $u \in \mathcal{U}(\mathbb{Z}[\zeta_{p^n}]^A)$ is of the type $\zeta_{p^n}^j$ or $\pm \zeta_{p^n}^j$ depending on p is even or odd with $0 \leq j < p^n$ satisfying

$$jx \equiv j \pmod{p^n} \quad \forall x \in A. \quad (8)$$

If p is odd, we have $A = \langle r \rangle$ for some $r \in \mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$ and if $p = 2$, then $A = \langle r \rangle$ or $\langle r \rangle \times \langle -1 \rangle$ for some $r \in \mathcal{U}(\mathbb{Z}/2^n\mathbb{Z})$.

Case(I): $A = \langle r \rangle$

We must have $o_{p^n}(r) = \varphi(p^n)$ or $\frac{\varphi(p^n)}{2}$.

Suppose $o_{p^n}(r) = \varphi(p^n)$. This case arises only if p is an odd prime or $p^n = 2$ or 4 . From Eq.(8), $-j \equiv j \pmod{p^n}$. This gives $j = 0$, 2^{n-1} , if $p^n = 2$ or 4 and $j = 0$, if p is an odd prime.

If $o_{p^n}(r) = \frac{\varphi(p^n)}{2}$, then Eq.(8) holds if and only if

$$jr \equiv j \pmod{p^n}. \quad (9)$$

Assuming $j \neq 0$, write $j = p^k j'$ with $p \nmid j'$ and $k \geq 0$. Eq.(9) implies that $r^{p^k} \equiv 1 \pmod{p^n}$ which gives that $\frac{\varphi(p^n)}{2}$ divides p^k . This is possible only if $p = 2$ or 3 . Furthermore, for $p = 2$, we have $n \geq 2$ and $j = 2^{n-2}, 2 \cdot 2^{n-2}, 3 \cdot 2^{n-2}$; and for

$p = 3$, we have $j = 3^{n-1}, 2 \cdot 3^{n-1}$.

Consequently,

$$\mathcal{U}(\mathbb{Z}[\zeta_{p^n}]^A) = \begin{cases} \{\zeta_{2^n}^0, \zeta_{2^n}^{2^{n-2}}, (\zeta_{2^n}^{2^{n-2}})^2, (\zeta_{2^n}^{2^{n-2}})^3\} = \langle \zeta_4 \rangle, & \text{if } r \equiv 1 \pmod{4}, n \geq 2; \\ \{\pm \zeta_{3^n}^0, \pm \zeta_{3^n}^{3^{n-1}}, \pm (\zeta_{3^n}^{3^{n-1}})^2\} = \langle \zeta_6 \rangle, & \text{if } r \equiv 1 \pmod{3}; \\ \{\pm 1\} = \langle \zeta_2 \rangle, & \text{otherwise.} \end{cases}$$

Case(II): $A = \langle r \rangle \times \langle -1 \rangle$

This case arises only if $p = 2$, $n \geq 3$ and Eq.(8) holds $\Leftrightarrow 2j \equiv 0 \pmod{2^n} \Leftrightarrow j = 0, 2^{n-1}$. This yields the desired result. \square

Proof of Theorem 2. (i) From ([15], Theorem 3.1), it follows immediately that $\text{rank of } \mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = 0$ if and only if $|I_i| = 1, \forall i, 1 \leq i \leq m$. Further, ([22], Corollary 7.3.3) implies that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm \mathcal{Z}(G)$ in this case.

(ii) Since $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ is a subgroup of $\prod_{i=1}^m \mathcal{A}(H_i, K_i)$, we have

$$\begin{aligned} [\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) : \langle \mathcal{B}(G) \rangle] &\leq [\prod_{i=1}^m \mathcal{A}(H_i, K_i) : \langle \mathcal{B}(G) \rangle] \\ &= [\prod_{i=1}^m \mathcal{A}(H_i, K_i) : \langle \cup_{i=1}^m \mathcal{B}(H_i, K_i) \rangle] \\ &= \prod_{i=1}^m [\mathcal{A}(H_i, K_i) : \langle \mathcal{B}(H_i, K_i) \rangle] \\ &= \prod_{i=1}^m [A(H_i, K_i) : \langle B(H_i, K_i) \rangle]. \end{aligned} \quad (10)$$

The last equality follows from Lemma 1. We now show that

$$[A(H_1, K_1) : \langle B(H_1, K_1) \rangle] \leq 2 \quad (11)$$

and for $2 \leq i \leq m$,

$$[A(H_i, K_i) : \langle B(H_i, K_i) \rangle] \leq \begin{cases} o_i, & \text{if } |I_i| = 1; \\ 2h_{p_i}^+ l_i p_i^{n_i-1} \mathfrak{o}_i(l_i^{d_i-1} [N_i : H_i])^{|I_i|-1}, & \text{if } |I_i| \neq 1. \end{cases} \quad (12)$$

Let $1 \leq i \leq m$. We have that the center of $\mathbb{Q}[N_i]\varepsilon_i$ is equal to $(\mathbb{Q}[H_i]\varepsilon_i)^{N_i/H_i}$, where $(\mathbb{Q}[H_i]\varepsilon_i)^{N_i/H_i}$ denotes the fixed field under the action of N_i/H_i on $\mathbb{Q}(\zeta_{p_i^{n_i}}) \cong \mathbb{Q}[H_i]\varepsilon_i$. Now, the center of $\mathbb{Q}(1 - \varepsilon_i) + \mathbb{Q}[N_i]\varepsilon_i$, which is equal to $\mathbb{Q}(1 - \varepsilon_i) + (\mathbb{Q}[H_i]\varepsilon_i)^{N_i/H_i}$, is embedded inside the algebra $\mathbb{Q}[H_i]\hat{K}_i \oplus \mathbb{Q}(1 - \hat{K}_i)$, via the embedding

$$r(1 - \varepsilon_i) + u \mapsto (r(1 - \varepsilon_i) + u)\hat{K}_i + r(1 - \hat{K}_i), \quad (13)$$

where $r \in \mathbb{Q}$ and $u \in (\mathbb{Q}[H_i]\varepsilon_i)^{N_i/H_i}$.

As $H_i/K_i = \langle g_i K_i \rangle$, any element $x\hat{K}_i \in \mathbb{Q}[H_i]\hat{K}_i$ can be expressed as $x\hat{K}_i = \sum_{j=0}^{p_i^{n_i}-1} x_j g_i^j \hat{K}_i$ with $x_j \in \mathbb{Q}$. Let π denote the projection of $\mathbb{Q}[H_i]\hat{K}_i \oplus \mathbb{Q}(1 - \hat{K}_i)$ onto $\mathbb{Q}(\zeta_{p_i^{n_i}})$ under the isomorphism $\mathbb{Q}[H_i]\hat{K}_i \oplus \mathbb{Q}(1 - \hat{K}_i) \xrightarrow{\tau} \oplus_{k=0}^{n_i} \mathbb{Q}(\zeta_{p_i^k}) \oplus \mathbb{Q}(1 - \hat{K}_i)$ given by

$$x\hat{K}_i + a(1 - \hat{K}_i) \mapsto \left(\sum_{j=0}^{p_i^{n_i}-1} x_j, \sum_{j=0}^{p_i^{n_i}-1} x_j \zeta_{p_i}^j, \dots, \sum_{j=0}^{p_i^{n_i}-1} x_j \zeta_{p_i^{n_i}}^j, a(1 - \hat{K}_i) \right), \quad (14)$$

where $x\hat{K}_i = \sum_{j=0}^{p_i^{n_i}-1} x_j g_i^j \hat{K}_i \in \mathbb{Q}[H_i]\hat{K}_i$ and $a \in \mathbb{Q}$.

Now observe that $\pi \circ \tau \circ \iota$ is injective on $\mathcal{Z}(\mathbb{Q}(1 - \varepsilon_i) + \mathbb{Q}[N_i]\varepsilon_i)$ and $\pi \circ \tau \circ \iota(A(H_i, K_i))$ is contained in $\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i})$. Hence,

$$\begin{aligned} [A(H_i, K_i) : \langle B(H_i, K_i) \rangle] &= [\pi \circ \tau \circ \iota(A(H_i, K_i)) : \pi \circ \tau \circ \iota(\langle B(H_i, K_i) \rangle)] \\ &\leq [\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) : \pi \circ \tau \circ \iota(\langle B(H_i, K_i) \rangle)]. \end{aligned} \quad (15)$$

If $i = 1$, i.e., $(H_i, K_i) = (G, G)$, then $[A(H_i, K_i) : \langle B(H_i, K_i) \rangle] \leq |\mathcal{U}(\mathbb{Z})| = 2$. Thus Eq.(11) holds.

If $2 \leq i \leq m$ is such that $|I_i| = 1$, then $B(H_i, K_i)$ is an empty set and therefore, $[\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) : \pi \circ \tau \circ \iota(\langle B(H_i, K_i) \rangle)] = |\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i})|$. We have using Lemma 2,

$$[A(H_i, K_i) : \langle B(H_i, K_i) \rangle] \leq o_i, \quad (16)$$

as $\langle N_i/H_i, -1 \rangle = \mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$ in this case.

We next assume that $|I_i| \neq 1$.

Set

$$\begin{aligned}
N_{(H_i, K_i)} &= \langle \pi_{N_i/H_i}(\eta_k(\zeta_{p_i^{n_i}})^{o_{p_i^{n_i}}(k)p_i^{n_i-1}n_{H_i, K_i}}) \mid k \in I_i \setminus \{1\} \rangle, \\
F_{(H_i, K_i)} &= \langle \eta_k(\zeta_{p_i^{n_i}})^{o_{p_i^{n_i}}(k)p_i^{n_i-1}n_{H_i, K_i}} \mid 1 < k < \frac{p_i^{n_i}}{2}, (k, p_i) = 1 \rangle, \\
O_{(H_i, K_i)} &= F_{(H_i, K_i)} \times \langle \zeta_{p_i^{n_i}}^{p_i^{n_i-1}}, -1 \rangle, \\
P_{(H_i, K_i)} &= \langle \eta_k(\zeta_{p_i^{n_i}}) \mid k \in \mathcal{U}(\mathbb{Z}/p_i^{n_i}\mathbb{Z}) \rangle = \langle \eta_k(\zeta_{p_i^{n_i}}) \mid 1 < k < \frac{p_i^{n_i}}{2}, (k, p_i) = 1 \rangle \times \langle \zeta_{p_i^{n_i}}, -1 \rangle, \\
Q_{(H_i, K_i)} &= \mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) \cap O_{(H_i, K_i)},
\end{aligned}$$

where $\pi_{N_i/H_i}(u) = \prod_{\sigma \in N_i/H_i} \sigma(u)$, for $u \in \mathbb{Q}(\zeta_{p_i^{n_i}})$.

By ([15], Proposition 3.4),

$$\pi \circ \tau \circ \iota(\langle B(H_i, K_i) \rangle) = N_{(H_i, K_i)}. \quad (17)$$

Therefore,

$$\begin{aligned}
[\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) : \pi \circ \tau \circ \iota(\langle B(H_i, K_i) \rangle)] &= [\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) : N_{(H_i, K_i)}] \\
&= [\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]^{N_i/H_i}) : Q_{(H_i, K_i)}][Q_{(H_i, K_i)} : N_{(H_i, K_i)}] \\
&\leq [\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]) : O_{(H_i, K_i)}][Q_{(H_i, K_i)} : N_{(H_i, K_i)}]. \quad (18)
\end{aligned}$$

Further,

$$[\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]) : O_{(H_i, K_i)}] = [\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]) : P_{(H_i, K_i)}][P_{(H_i, K_i)} : O_{(H_i, K_i)}] \quad (19)$$

Clearly,

$$[P_{(H_i, K_i)} : O_{(H_i, K_i)}] = p_i^{n_i-1} \prod_{\substack{1 < k < \frac{p_i^{n_i}}{2} \\ (k, p_i) = 1}} o_{p_i^{n_i}}(k) p_i^{n_i-1} n_{H_i, K_i} = p_i^{n_i-1} \mathfrak{o}_i. \quad (20)$$

Also, by ([23], Theorem 8.2),

$$[\mathcal{U}(\mathbb{Z}[\zeta_{p_i^{n_i}}]) : P_{(H_i, K_i)}] = h_{p_i^{n_i}}^+. \quad (21)$$

Next, observe that $Q_{(H_i, K_i)} \cap F_{(H_i, K_i)}$ is a free abelian group, and by ([15], Lemma 3.2), it has rank at most $|I_i| - 1$. Furthermore, any element of $Q_{(H_i, K_i)} \cap F_{(H_i, K_i)}$ is of order at most $l_i^{d_i-1} |N_i/H_i|$ modulo $N_{(H_i, K_i)} \cap F_{(H_i, K_i)}$. To see this, let $u \in Q_{(H_i, K_i)} \cap F_{(H_i, K_i)}$ and write $u = \prod_{\substack{1 < k < \frac{p_i^{n_i}}{2} \\ (k, p_i) = 1}} (\eta_k(\zeta_{p_i^{n_i}})^{o_{p_i^{n_i}}(k)p_i^{n_i-1}n_{H_i, K_i}})^{\alpha_k}$, $\alpha_k \geq 0$. Since $\pi_{N_i/H_i}(\eta_{r_i^t}(\zeta_{p_i^{n_i}}^j)) = 1$ and $\pi_{N_i/H_i}(\eta_{-j}(\zeta_{p_i^{n_i}})) = \pi_{N_i/H_i}(-\zeta_{p_i^{n_i}}^{-j})\pi_{N_i/H_i}(\eta_j(\zeta_{p_i^{n_i}}))$, for $i, j \geq 0$, it turns out that $u^{|N_i/H_i|l_i^{d_i-1}} = (\pi_{N_i/H_i}(u))^{l_i^{d_i-1}} \in N_{(H_i, K_i)} \cap F_{(H_i, K_i)}$.

Consequently,

$$[Q_{(H_i, K_i)} \cap F_{(H_i, K_i)} : N_{(H_i, K_i)} \cap F_{(H_i, K_i)}] \leq (l_i^{d_i-1} |N_i/H_i|)^{|I_i|-1} \quad (22)$$

and therefore,

$$\begin{aligned} [Q_{(H_i, K_i)} : N_{(H_i, K_i)}] &\leq [Q_{(H_i, K_i)} : N_{(H_i, K_i)} \cap F_{(H_i, K_i)}] \\ &= [Q_{(H_i, K_i)} : Q_{(H_i, K_i)} \cap F_{(H_i, K_i)}][Q_{(H_i, K_i)} \cap F_{(H_i, K_i)} : N_{(H_i, K_i)} \cap F_{(H_i, K_i)}] \\ &\leq l_i(l_i^{d_i-1} |N_i/H_i|)^{|I_i|-1}. \end{aligned} \quad (23)$$

Finally, Eqs.(15)-(23) yield Eq.(12), which in view of Eq.(10) and Eq.(11) complete the proof. \square

It is known that if G is an abelian group, then $\mathcal{U}(\mathbb{Z}[G]) = \pm G$ if and only if G is of exponent 1, 2, 3, 4 or 6 (see [21], Theorem 2.7). We have the following:

Corollary 1 *Let G be a strongly monomial group with a complete irredundant set $\{(H_i, K_i) \mid 1 \leq i \leq m\}$ of strong Shoda pairs such that $[H_i : K_i] = 1, 2, 3, 4$ or 6 for all $i, 1 \leq i \leq m$. Then, $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm \mathcal{Z}(G)$.*

In particular, if G is a strongly monomial (e.g. abelian by supersolvable) group of exponent 1, 2, 3, 4 or 6, then $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm \mathcal{Z}(G)$. However, the converse need not be true.

Proof. Here, $|I_i| = 1$ for all $i, 1 \leq i \leq m$. Therefore, Theorem 2(i) is applicable. The group \mathcal{G}_1 defined in section 4.1 for $p = 3$ is an example of a strongly monomial group of exponent 9 satisfying $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm \mathcal{Z}(G)$. \square

For abelian p -groups, Theorem 2 gives the following:

Corollary 2 *Let G be an abelian p -group, p prime, and let $K_i, 1 \leq i \leq m$, be all the subgroups of G with cyclic quotient groups. Suppose $[G : K_i] = p^{n_i}$, for $1 \leq i \leq m$. Then, the rank of $\mathcal{U}(\mathbb{Z}[G])$ is non zero if and only if $p^{n_i} \geq 5$ for some i . In this case, the index of $\langle \mathcal{B}(G) \rangle$ in $\mathcal{U}(\mathbb{Z}[G])$ is at most*

$$2 \prod_{i=1}^m 2^{n_i} h_{2^{n_i}}^+ \left(\prod_{\substack{1 \leq k < 2^{n_i}-1 \\ (k,2)=1}} 2^{n_i} o_{2^{n_i}}(k) n_{G, K_i} \right), \text{ if } p = 2.$$

and

$$\prod_{i=1}^m 2p^{n_i} h_{p^{n_i}}^+ \left(\prod_{\substack{1 \leq k < \frac{p^{n_i}}{2} \\ (k,p)=1}} 2p^{n_i} o_{p^{n_i}}(k) n_{G, K_i} \right), \text{ if } p \neq 2;$$

where an empty product equals 1.

4 Non Abelian groups of order p^n , $n \leq 4$

Let G be a non abelian group of order p^n , $n \leq 4$. Observe that any such group, being metabelian, is normally monomial.

4.1 Non Abelian groups of order p^3

If $p = 2$, then G is either isomorphic to D_4 , the dihedral group of order 8 or is isomorphic to Q_8 , the group of quaternions. Both these groups satisfy the hypothesis of ([21], Theorem 6.1). Therefore, we already know that the group of central units in the integral group ring of these groups consists of only the trivial units.

If p is an odd prime, then G is isomorphic to one of the following groups:

- $\mathcal{G}_1 = \langle a, b \mid a^{p^2} = b^p = 1, ab = ba^{p+1} \rangle$;
- $\mathcal{G}_2 = \langle a, b, c \mid a^p = b^p = c^p = 1, ab = bac, ac = ca, bc = cb \rangle$.

In ([3], Theorems 3 and 4), a complete and irredundant set of strong Shoda pairs of these groups has been found. Applying Theorem 3.1 of [15], we obtain that

$$\text{Rank of } \mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{G}_i])) = \frac{(p-3)(p+2)}{2}, \quad i = 1, 2.$$

We now illustrate Theorem 2 in the particular cases, when $p = 3$ or 5.

p=3 : In this case, the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{G}_i])) = 0$ and therefore, by Theorem 2(i), $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{G}_i])) = \pm \mathcal{Z}(\mathcal{G}_i)$, $i = 1, 2$.

p=5 : In this case, the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{G}_i])) = 7$, $i = 1, 2$. Using Remark 1, we have computed the value of $n_{H,K}$ corresponding to each strong Shoda pair (H, K) of the groups \mathcal{G}_1 and \mathcal{G}_2 , which are tabulated below:

(H, K)	$n_{H,K}$	(H, K)	$n_{H,K}$
$(\mathcal{G}_1, \mathcal{G}_1)$	1	$(\mathcal{G}_2, \mathcal{G}_2)$	1
$(\langle a^5, b \rangle, \langle b \rangle)$	5	$(\langle a, c \rangle, \langle a \rangle)$	5
$(\mathcal{G}_1, \langle a \rangle)$	5^2	$(\mathcal{G}_2, \langle b, c \rangle)$	5^2
$(\mathcal{G}_1, \langle a^5, b \rangle)$	5^2	$(\mathcal{G}_2, \langle a, c \rangle)$	5^2
$(\mathcal{G}_1, \langle ab \rangle)$	5^2	$(\mathcal{G}_2, \langle ab, c \rangle)$	5^2
$(\mathcal{G}_1, \langle a^2b \rangle)$	5^2	$(\mathcal{G}_2, \langle a^2b, c \rangle)$	5^2
$(\mathcal{G}_1, \langle a^3b \rangle)$	5^2	$(\mathcal{G}_2, \langle a^3b, c \rangle)$	5^2
$(\mathcal{G}_1, \langle a^4b \rangle)$	5^2	$(\mathcal{G}_2, \langle a^4b, c \rangle)$	5^2

Fig.1 Strong Shoda pairs of \mathcal{G}_1

Fig.2 Strong Shoda pairs of \mathcal{G}_2

Theorem 2 and ([23], §11.5) yield that $[\mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{G}_i])) : \langle \mathcal{B}(\mathcal{G}_i) \rangle] \leq 2^{29}5^{27}$, $i = 1, 2$.

4.2 Non Abelian groups of order p^4

We first take the case, when $p = 2$. Upto isomorphism, there are 9 non isomorphic groups of order 2^4 as listed in ([5], §118). Except the following two groups:

- $\mathcal{H}_1 = \langle a, b : a^8 = b^2 = 1, ba = a^7b \rangle;$
- $\mathcal{H}_2 = \langle a, b : a^8 = b^4 = 1, ba = a^7b, a^4 = b^2 \rangle,$

the other non abelian groups of order 2^4 again satisfy the hypothesis of ([21], Theorem 6.1). Hence, if G is a non abelian group of order 2^4 other than \mathcal{H}_1 and \mathcal{H}_2 , then $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm \mathcal{Z}(G)$.

For the groups \mathcal{H}_1 and \mathcal{H}_2 , we obtain using Theorem 1, that $\{(\langle a \rangle, \langle 1 \rangle), (\langle a \rangle, \langle a^4 \rangle), (\mathcal{H}_1, \langle a \rangle), (\mathcal{H}_1, \langle a^2, b \rangle), (\mathcal{H}_1, \langle a^2, ab \rangle), (\mathcal{H}_1, \mathcal{H}_1)\}$ and $\{(\langle a \rangle, \langle 1 \rangle), (\langle a \rangle, \langle a^4 \rangle), (\mathcal{H}_2, \langle a \rangle), (\mathcal{H}_2, \langle a^2, b \rangle), (\mathcal{H}_2, \langle a^2, ab \rangle), (\mathcal{H}_2, \mathcal{H}_2)\}$ are complete irredundant sets of strong Shoda pairs of \mathcal{H}_1 and \mathcal{H}_2 respectively. Theorem 3.1 of [15] now yields that

$$\text{Rank of } \mathcal{Z}(\mathcal{U}(\mathbb{Z}[H_i])) = 1, \quad i = 1, 2.$$

Also, by Theorem 2 and ([23], §11.5), it follows that

$$[\mathcal{Z}(\mathcal{U}(\mathbb{Z}[\mathcal{H}_i])) : \langle \mathcal{B}(\mathcal{H}_i) \rangle] \leq 2^{12}, \quad i = 1, 2.$$

We next assume that p is an odd prime.

Upto isomorphism, the following are all the non abelian groups of order p^4 (see [5], §117):

1. $G_1 = \langle a, b : a^{p^3} = b^p = 1, ba = a^{1+p^2}b \rangle;$
2. $G_2 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, cb = a^pbc, ab = ba, ac = ca \rangle;$
3. $G_3 = \langle a, b : a^{p^2} = b^{p^2} = 1, ba = a^{1+p}b \rangle;$
4. $G_4 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, ca = a^{1+p}c, ba = ab, cb = bc \rangle;$
5. $G_5 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, ca = abc, ab = ba, bc = cb \rangle;$
6. $G_6 = \langle a, b, c : a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = abc, cb = bc \rangle;$
7. $G_7 = \begin{cases} \langle a, b, c : a^{p^2} = b^p = 1, c^p = a^p, ab = ba^{1+p}, ac = cab^{-1}, cb = bc \rangle, & \text{if } p = 3, \\ \langle a, b, c : a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = a^{1+p}bc, cb = a^pbc \rangle, & \text{if } p > 3; \end{cases}$
8. $G_8 = \begin{cases} \langle a, b, c : a^{p^2} = b^p = 1, c^p = a^{-p}, ab = ba^{1+p}, ac = cab^{-1}, cb = bc \rangle, & \text{if } p = 3, \\ \langle a, b, c : a^{p^2} = b^p = c^p = 1, ba = a^{1+p}b, ca = a^{1+dp}bc, cb = a^{dp}bc \rangle, & \text{if } p > 3 \\ d \not\equiv 0, 1 \pmod{p}; \end{cases}$
9. $G_9 = \langle a, b, c, d : a^p = b^p = c^p = d^p = 1, dc = acd, bd = db, ad = da, bc = cb, ac = ca, ab = ba \rangle;$
10. $G_{10} = \begin{cases} \langle a, b, c : a^{p^2} = b^p = c^p = 1, ab = ba, ac = cab, bc = ca^{-p}b \rangle, & \text{if } p = 3, \\ \langle a, b, c, d : a^p = b^p = c^p = d^p = 1, dc = bcd, db = abd, ad = da, bc = cb, ac = ca, ab = ba \rangle, & \text{if } p > 3. \end{cases}$

Theorem 3 *The set $S(G_i)$ of strong Shoda pairs for each G_i , $1 \leq i \leq 10$, is as follows:*

- (i) $\mathcal{S}(G_1) = \{(\langle a \rangle, \langle 1 \rangle), (G_1, \langle a \rangle), (G_1, G_1)\} \cup \{(G_1, \langle a^{p^2}, a^{pi}b \rangle), (G_1, \langle a^p, a^ib \rangle) \mid 0 \leq i \leq p-1\};$
- (ii) $\mathcal{S}(G_2) = \{(\langle a, b \rangle, \langle b \rangle), (G_2, \langle a, b \rangle), (G_2, G_2)\} \cup \{(G_2, \langle a, b^ic \rangle), (G_2, \langle a^ib, a^jc \rangle) \mid 0 \leq i, j \leq p-1\};$
- (iii) $\mathcal{S}(G_3) = \{(G_3, \langle a, b^p \rangle), (G_3, \langle a \rangle), (G_3, G_3)\} \cup \{(\langle a, b^p \rangle, \langle a^{pi}b^p \rangle), (G_3, \langle a^p, a^ib \rangle) \mid 0 \leq i \leq p-1\} \cup \{(G_3, \langle a^p, a^kb^p \rangle) \mid 1 \leq k \leq p-1\};$
- (iv) $\mathcal{S}(G_4) = \{(G_4, \langle a, b \rangle), (G_4, G_4)\} \cup \{(\langle a, b \rangle, \langle a^{pi}b \rangle), (G_4, \langle a, b^ic \rangle), (G_4, \langle a^p, a^ib, a^jc \rangle) \mid 0 \leq i, j \leq p-1\};$

$$(v) \mathcal{S}(G_5) = \{(\langle a, b \rangle, \langle a \rangle), (G_5, \langle a^p, b, c \rangle), (G_5, \langle a, b \rangle), (G_5, G_5)\} \cup \\ \{(G_5, \langle b, a^{pi}c \rangle) \mid 0 \leq i \leq p-1\} \cup \\ \{(\langle a, b \rangle, \langle a^p b^k \rangle), (G_5, \langle b, a^k c \rangle) \mid 1 \leq k \leq p-1\};$$

$$(vi) \mathcal{S}(G_6) = \{(\langle a^p, b, c \rangle, \langle a^p, c \rangle), (G_6, \langle a, b \rangle), (G_6, \langle a^p, b, c \rangle), (G_6, G_6)\} \cup \\ \{(\langle a^p, b, c \rangle, \langle b, a^{pi}c \rangle) \mid 0 \leq i \leq p-1\} \cup \\ \{(G_6, \langle b, a^k c \rangle) \mid 1 \leq k \leq p-1\};$$

$$(vii) \mathcal{S}(G_7) = \{(\langle b, c \rangle, \langle b \rangle), (\langle b, c \rangle, \langle c \rangle), (G_7, \langle a, b \rangle), (G_7, G_7)\} \cup \\ (p=3) \quad \{(G_7, \langle b, a^i c \rangle) \mid 0 \leq i \leq p-1\};$$

$$(viii) \mathcal{S}(G_7) = \{(\langle b, ac \rangle, \langle b \rangle), (\langle b, ac \rangle, \langle ac \rangle), (G_7, \langle a, b \rangle), (G_7, G_7)\} \cup \\ (p>3) \quad \{(G_7, \langle b, a^i c \rangle) \mid 0 \leq i \leq p-1\};$$

$$(ix) \mathcal{S}(G_8) = \{(\langle b, c \rangle, \langle b \rangle), (\langle b, c \rangle, \langle c \rangle), (G_8, \langle a, b \rangle), (G_8, G_8)\} \cup \\ (p=3) \quad \{(G_8, \langle b, a^i c \rangle) \mid 0 \leq i \leq p-1\};$$

$$(x) \mathcal{S}(G_8) = \{(\langle b, a^d c \rangle, \langle b \rangle), (\langle b, a^d c \rangle, \langle a^d c \rangle), (G_8, \langle a, b \rangle), (G_8, G_8)\} \cup \\ (p>3) \quad \{(G_8, \langle b, a^i c \rangle) \mid 0 \leq i \leq p-1\};$$

$$(xi) \mathcal{S}(G_9) = \{(G_9, \langle a, b, d \rangle), (G_9, G_9)\} \cup \\ \{(\langle a, b, d \rangle, \langle d, a^i b \rangle), (G_9, \langle a, b, cd^i \rangle), (G_9, \langle a, b^i c, b^j d \rangle) \mid 0 \leq i, j \leq p-1\};$$

$$(xii) \mathcal{S}(G_{10}) = \{(\langle a, b \rangle, \langle a \rangle), (\langle a, b \rangle, \langle b \rangle), (G_{10}, \langle a, b \rangle), (G_{10}, G_{10})\} \cup \\ (p=3) \quad \{(G_{10}, \langle b, a^i c \rangle) \mid 0 \leq i \leq p-1\};$$

$$(xiii) \mathcal{S}(G_{10}) = \{(\langle a, b, c \rangle, \langle a, c \rangle), (G_{10}, \langle a, b, d \rangle)\} \cup \\ (p>3) \quad \{(\langle a, b, c \rangle, \langle a^i c, b \rangle), (G_{10}, \langle a, b, cd^i \rangle), (G_{10}, G_{10}) \mid 0 \leq i \leq p-1\}.$$

Proof. (i) Define $N_0 := \langle 1 \rangle$, $N_1 := \langle a^{p^2} \rangle$, $N_2 := \langle a^p \rangle$, $N_3 := \langle a \rangle$, $H_i := \langle a^{p^2}, a^{pi}b \rangle$, $K_j := \langle a^p, a^j b \rangle$ where $0 \leq i, j \leq p-1$. Observe that these subgroups are normal in G_1 . Using Eq.(1), we have $\mathcal{S}_{N_1} = \mathcal{S}_{N_2} = \phi$, $\mathcal{S}_{N_3} = \{(G_1, N_3)\}$, $\mathcal{S}_{H_i} = \{(G_1, H_i)\}$, $\mathcal{S}_{K_j} = \{(G_1, K_j)\}$, $0 \leq i, j \leq p-1$. In order to find \mathcal{S}_{N_0} , we see that $\langle a \rangle$ is a maximal abelian subgroup of G_1 . Further, the only subgroup D of $\langle a \rangle$ satisfying $\text{core}(D) = \langle 1 \rangle$ is $D = \langle 1 \rangle$. This gives $\mathcal{S}_{N_0} = \{(\langle a \rangle, \langle 1 \rangle)\}$. Define

$$\mathcal{N}_1 = \{\langle 1 \rangle, \langle a^{p^2} \rangle, \langle a^p \rangle, \langle a \rangle, \langle a, b \rangle\} \cup \{\langle a^{p^2}, a^{pi}b \rangle, \langle a^p, a^j b \rangle \mid 0 \leq i, j \leq p-1\}.$$

Observe that $\sum_{N \in \mathcal{N}_1} \sum_{D \in \mathcal{D}_N} [G : A_N] \varphi([A_N : D]) = p^4$. Now, if \mathcal{N} is the set of all normal subgroups of G_1 , then

$$\begin{aligned}
p^4 = |G_1| &= \sum_{N \in \mathcal{N}} \sum_{D \in \mathcal{D}_N} [G : A_N] \varphi([A_N : D]) \quad (\text{by Theorem 1}) \\
&\geq \sum_{N \in \mathcal{N}_1} \sum_{D \in \mathcal{D}_N} [G : A_N] \varphi([A_N : D]) \quad (\text{as } \mathcal{N}_1 \subseteq \mathcal{N}) \\
&= p^4.
\end{aligned}$$

This yields $\mathcal{S}_N = \phi$, if $N \notin \mathcal{N}_1$ and consequently, by Theorem 1, $\bigcup_{N \in \mathcal{N}_1} \mathcal{S}_N$ is a complete irredundant set of strong Shoda pairs of G_1 .

(ii)-(xiii) For $2 \leq i \leq 10$, consider the following set \mathcal{N}_i of normal subgroups of G_i :

$$\begin{aligned}
\mathcal{N}_2 &= \{ \langle 1 \rangle, \langle a^p \rangle, \langle a^p, b \rangle, \langle a, b \rangle, \langle a, b, c \rangle \} \cup \\
&\quad \{ \langle a^p, b^i c \rangle, \langle a, b^i c \rangle, \langle ab^i c^j \rangle, \langle a^i b, a^j c \rangle \mid 0 \leq i, j \leq p-1 \}; \\
\mathcal{N}_3 &= \{ \langle 1 \rangle, \langle a^p \rangle, \langle a \rangle, \langle a, b^p \rangle, \langle a^p, b^p \rangle, \langle a, b \rangle \} \cup \\
&\quad \{ \langle a^{pi} b^p \rangle, \langle a^p, a^i b \rangle \mid 0 \leq i \leq p-1 \} \cup \{ \langle a^p, a^k b^p \rangle \mid 1 \leq k \leq p-1 \}; \\
\mathcal{N}_4 &= \{ \langle 1 \rangle, \langle a^p \rangle, \langle a^p, b \rangle, \langle a, b \rangle, \langle a, b, c \rangle \} \cup \\
&\quad \{ \langle a^{pi} b \rangle, \langle a^p, b^i c \rangle, \langle a, b^i c \rangle, \langle ab^i c^j \rangle, \langle a^p, a^i b, a^j c \rangle \mid 0 \leq i, j \leq p-1 \}; \\
\mathcal{N}_5 &= \{ \langle 1 \rangle, \langle b \rangle, \langle a^p, b \rangle, \langle a^p \rangle, \langle a^p, b, c \rangle, \langle a, b \rangle, \langle a, b, c \rangle \} \cup \\
&\quad \{ \langle b, a^{pi} c \rangle \mid 0 \leq i \leq p-1 \} \cup \{ \langle a^p b^k \rangle, \langle b, a^k c \rangle \mid 1 \leq k \leq p-1 \}; \\
\mathcal{N}_6 &= \{ \langle 1 \rangle, \langle a^p \rangle, \langle a^p, b \rangle, \langle a, b \rangle, \langle a^p, b, c \rangle, \langle a, b, c \rangle \} \cup \{ \langle b, a^k c \rangle \mid 1 \leq k \leq p-1 \}; \\
\mathcal{N}_7 &= \{ \langle 1 \rangle, \langle a^p \rangle, \langle a^p, b \rangle, \langle a, b \rangle, \langle a, b, c \rangle \} \cup \{ \langle b, a^i c \rangle, \mid 0 \leq i \leq p-1 \}; \\
\mathcal{N}_8 &= \{ \langle 1 \rangle, \langle a^p \rangle, \langle a^p, b \rangle, \langle a, b \rangle, \langle a, b, c \rangle \} \cup \{ \langle b, a^i c \rangle, \mid 0 \leq i \leq p-1 \}; \\
\mathcal{N}_9 &= \{ \langle 1 \rangle, \langle a \rangle, \langle a, d \rangle, \langle a, b, d \rangle, \langle a, b, c, d \rangle \} \cup \\
&\quad \{ \langle a^i b \rangle, \langle a, bc^i d^j \rangle, \langle a, cd^i \rangle, \langle a, b, cd^i \rangle, \langle a, b^i c, b^j d \rangle \mid 0 \leq i, j \leq p-1 \}; \\
\mathcal{N}_{10} &= \begin{cases} \{ \langle 1 \rangle, \langle a^3 \rangle, \langle a^3, b \rangle, \langle b, c \rangle, \langle b, ac \rangle, \langle b, a^2 c \rangle, \langle a, b \rangle, \langle a, b, c \rangle \}, & \text{if } p = 3; \\ \{ \langle 1 \rangle, \langle a \rangle, \langle a, b \rangle, \langle a, b, d \rangle, \langle a, b, c, d \rangle \} \cup \{ \langle a, b, cd^i \rangle \mid 0 \leq i \leq p-1 \}, & \text{if } p > 3. \end{cases}
\end{aligned}$$

Now proceeding as in (i), we get the required complete and irredundant set of strong Shoda pairs of G_i , $2 \leq i \leq 10$. \square

For a particular odd prime p , the computation of $n_{H,K}$ corresponding to a strong Shoda pair $(H, K) \in \mathcal{S}(G_i)$, $1 \leq i \leq 10$, may be done using Remark 1. An explicit bound on the index of $\langle \mathcal{B}(G_i) \rangle$ in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G_i]))$, $1 \leq i \leq 10$, may thus be computed using Theorems 2 and 3.

Remark 2 *It would be of interest to compute the integer $n_{H,K}$ corresponding to each strong Shoda pairs (H, K) of the groups discussed in this section, explicitly in terms of p .*

Finally, Theorem 3 along with ([19], Proposition 3.4) and ([15], Theorem 3.1) also yield the following:

Corollary 3 *The Wedderburn decomposition of $\mathbb{Q}[G_i]$ and the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G_i]))$, $1 \leq i \leq 10$, are as follows:*

G	$\mathbb{Q}[G]$	Rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$
G_1	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus \mathbb{Q}(\zeta_{p^2})^{(p)} \oplus M_p(\mathbb{Q}(\zeta_{p^2}))$	$\frac{(p+1)(p^2-5)}{2}$
G_2	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p+p^2)} \oplus M_p(\mathbb{Q}(\zeta_{p^2}))$	$\frac{p^3-p^2-3p-5}{2}$
G_3	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus \mathbb{Q}(\zeta_{p^2})^{(p)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(p)}$	$\frac{p^3+p^2-7p-3}{2}$
G_4	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p+p^2)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(p)}$	$\frac{(p-3)(p+1)^2}{2}$
G_5	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus \mathbb{Q}(\zeta_{p^2})^{(p)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(p)}$	$\frac{p^3+p^2-7p-3}{2}$
G_6	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(1+p)}$	$(p-3)(p+1)$
G_7	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p)) \oplus M_p(\mathbb{Q}(\zeta_{p^2}))$	$p^2 - p - 4$
G_8	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p)) \oplus M_p(\mathbb{Q}(\zeta_{p^2}))$	$p^2 - p - 4$
G_9	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p+p^2)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(p)}$	$\frac{(p-3)(p+1)^2}{2}$
G_{10} ($p = 3$)	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_3)^{(4)} \oplus M_3(\mathbb{Q}(\zeta_3)) \oplus M_3(\mathbb{Q}(\zeta_9))$	2
G_{10} ($p > 3$)	$\mathbb{Q} \oplus \mathbb{Q}(\zeta_p)^{(1+p)} \oplus M_p(\mathbb{Q}(\zeta_p))^{(1+p)}$	$(p-3)(p+1)$

Acknowledgements

The authors are grateful to the anonymous referees for their valuable comments and suggestions which has helped to write the paper in the present form.

References

- [1] R. Ž. Aleev, *Higman's central unit theory, units of integral group rings of finite cyclic groups and Fibonacci numbers*, Internat. J. Algebra Comput. **4** (1994), no. 3, 309–358.

- [2] R. Zh. Aleev and G. A. Panina, *The units of cyclic groups of orders 7 and 9*, Izv. Vyssh. Uchebn. Zaved. Mat. (1999), no. 11, 81–84.
- [3] G.K. Bakshi and S. Maheshwary, *The rational group algebra of a normally monomial group*, J. Pure Appl. Algebra **218** (2014), no. 9, 1583–1593.
- [4] H. Bass, *The Dirichlet unit theorem, induced characters, and Whitehead groups of finite groups*, Topology **4** (1965), 391–410.
- [5] W. Burnside, *Theory of groups of finite order*, Dover Publications, Inc., New York, 1955, 2d ed.
- [6] R. A. Ferraz and J. J. Simón-Pinero, *Central units in metacyclic integral group rings*, Comm. Algebra **36** (2008), no. 10, 3708–3722.
- [7] Raul Antonio Ferraz, *Units of $\mathbb{Z}C_p$* , Groups, rings and group rings, Contemp. Math., vol. 499, Amer. Math. Soc., Providence, RI, 2009, pp. 107–119.
- [8] J. Z. Gonçalves and D. S. Passman, *Linear groups and group rings*, J. Algebra **295** (2006), no. 1, 94–118.
- [9] K. Hoechsmann, *Constructing units in commutative group rings*, Manuscripta Math. **75** (1992), no. 1, 5–23.
- [10] ———, *Unit bases in small cyclic group rings*, Methods in ring theory (Levico Terme, 1997), Lecture Notes in Pure and Appl. Math., vol. 198, Dekker, New York, 1998, pp. 121–139.
- [11] K. Hoechsmann and S. K. Sehgal, *Units in regular abelian p -group rings*, J. Number Theory **30** (1988), no. 3, 375–381.
- [12] E. Jespers, Á. del Río, and I. Van Gelder, *Writing units of integral group rings of finite abelian groups as a product of Bass units*, Math. Comp. **83** (2014), no. 285, 461–473.
- [13] E. Jespers, Á. Olteanu, G. and del Río, and I. Van Gelder, *Central units of integral group rings*, Proc. Amer. Math Soc. **142** (2014), 2193–2209.
- [14] E. Jespers, G. Olteanu, and Á. del Río, *Rational group algebras of finite groups: from idempotents to units of integral group rings*, Algebr. Represent. Theory **15** (2012), no. 2, 359–377.

- [15] E. Jespers, G. Olteanu, Á. del Río, and I. Van Gelder, *Group rings of finite strongly monomial groups: central units and primitive idempotents*, J. Algebra **387** (2013), 99–116.
- [16] E. Jespers and M. M. Parmenter, *Construction of central units in integral group rings of finite groups*, Proc. Amer. Math. Soc. **140** (2012), no. 1, 99–107.
- [17] E. Jespers, M. M. Parmenter, and S. K. Sehgal, *Central units of integral group rings of nilpotent groups*, Proc. Amer. Math. Soc. **124** (1996), no. 4, 1007–1012.
- [18] Y. Li and M. M. Parmenter, *Central units of the integral group ring $\mathbf{Z}A_5$* , Proc. Amer. Math. Soc. **125** (1997), no. 1, 61–65.
- [19] A. Olivieri, Á. del Río, and J. J. Simón, *On monomial characters and central idempotents of rational group algebras*, Comm. Algebra **32** (2004), no. 4, 1531–1550.
- [20] C. Polcino Milies and S. K. Sehgal, *An introduction to group rings*, Algebras and Applications, vol. 1, Kluwer Academic Publishers, Dordrecht, 2002.
- [21] S. K. Sehgal, *Units in integral group rings*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 69, Longman Scientific & Technical, Harlow, 1993, With an appendix by Al Weiss.
- [22] Sudarshan K. Sehgal, *Topics in group rings*, Monographs and Textbooks in Pure and Applied Math., vol. 50, Marcel Dekker, Inc., New York, 1978.
- [23] L. C. Washington, *Introduction to cyclotomic fields*, second ed., Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, 1997.